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Translated by L.K.

UDC 531.36

ON THE STABILITY OF THE EQUILIBRIUM POSITIONS OF A SOLID BODY WITH A CAVITY CONTAINING A LIQUID

PMM Vol. 36, №5, 1972, pp. 822-828
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(Received January 5, 1972)

We consider a solid body with a simply-connected cavity containing a liquid. In the case when the potential energy is positive definite with respect to a part of the generalized coordinates, we give sufficient conditions for the asymptotic stability of the equilibrium position relative to a part of the coordinates, to the generalized velocities, and to the kinetic energy of the fluid. It is shown that the asymptotic stability is uniform with respect to initial excitations from any compact set in some neighborhood of the equilibrium position.

1. We consider a system of differential equations of perturbed motion

$$\mathbf{x}^{\cdot} = \mathbf{X}(t, \mathbf{x}) \qquad (\mathbf{X}(t, 0) \equiv 0) \tag{1.1}$$

where $\mathbf{x} = (y_1, \ldots, y_m, \quad z_1, \ldots, z_p)$ is a real n-vector and, $n = m + p$, $m > 0$, $p \geqslant 0$. We assume that

a) in the region

$$t \geqslant 0, \quad \|\mathbf{y}\| \leqslant H > 0, \quad 0 \leqslant \|\mathbf{z}\| < +\infty \tag{1.2}$$

the right-hand sides of system (1.1) are continuous and satisfy the conditions for the uniqueness of the solution;

b) the solutions of system (1.1) are z-continuable, i.e. esch solution $\mathbf{x}(t)$ is defined for all $t \geqslant 0$ for which $\|\mathbf{y}(t)\| \leqslant H$.

The solution of system (1.1), determined by the initial condition $\mathbf{x}(t_0;\ t_0,\ \mathbf{x}_0) = \mathbf{x}_0$, is denoted $\mathbf{x} = \mathbf{x}(t;\ t_0,\ \mathbf{x}_0)$. Here we adopt the notation of the survey paper [1].

Theorem 1. If a function $V(t,\ \mathbf{x})$ exists, satisfying in region (1.2) the conditions:

1) $V(t,\ \mathbf{x}) \geqslant a(\|\mathbf{y}\|)$, where $a(r)$ is a continuous function, monotonically increasing on $[0,\ H]$, $a(0) = 0$;

2) $V^{\bullet}(t,\ \mathbf{x}) \leqslant -W(t,\ \mathbf{x})$ and $W(t,\ \mathbf{x}) \geqslant b(\|\mathbf{y}\|)$ ($b(r)$ is a function of the type of $a(r)$);

3) for any $t_0 \geqslant 0$ we can find $\delta'(t_0) > 0$ such that for each point $\mathbf{x}_0$ with $\|\mathbf{x}_0\| < \delta'$ there exists a constant $M(t_0,\ \mathbf{x}_0) > 0$ which bounds $W^{\bullet}$ along every solution starting in some neighborhood of the point $\mathbf{x} = 0$, i.e.

$$|W^{\bullet}(t, \mathbf{x}(t; t_0, \mathbf{x}_0))| \leqslant M \quad \text{for} \quad t \geqslant t_0 \tag{1.3}$$

then the motion $\mathbf{x} = 0$ is asymptotically y-stable.

Proof. The motion $\mathbf{x} = 0$ is y-stable by virtue of conditions (1) and (2) [2], therefore, for any $\varepsilon > 0$, $t_0 \geqslant 0$ there exists $\delta(\varepsilon, t_0)$, $0 < \delta < \delta'$, such that from $\|\mathbf{x}_0\| < \delta$ follows $\|\mathbf{y}(t;\ t_0, \mathbf{x}_0)\| < \varepsilon$ for all $t \geqslant t_0$. Let us show that $\lim W(t, \mathbf{x}(t;\ t_0, \mathbf{x}_0)) = 0$ as $t \to \infty$ if only $\|\mathbf{x}_0\| < \delta$. We assume the contrary: let there exist a number $l > 0$, a point $\mathbf{x}_*$ with $\|\mathbf{x}_*\| < \delta$, and a sequence $t_k \to \infty$ $(k = 1, 2, 3, \ldots)$, $t_k - t_{k-1} \geqslant \alpha > 0$, for which

$$W(t_k), \mathbf{x}(t_k, t_0, \mathbf{x}_*)) \geqslant l \tag{1.4}$$

On the basis of (1.3) and (1.4), from the relation

$$W(t, \mathbf{x}(t; t_0, \mathbf{x}_*)) = W(t_k, \mathbf{x}(t_k; t_0, \mathbf{x}_*)) + \int_{t_k}^{t} W^{\bullet}(\tau, \mathbf{x}(\tau; t_0, \mathbf{x}_*))\, d\tau$$

we conclude that there exists β, $0 < \beta < \alpha / 2$, for which $W(t, \mathbf{x}(t;\ t_0, \mathbf{x}_*)) \geqslant l / 2$ for $t \in [t_k - \beta,\ t_k + \beta]$ for all $k = 1, 2, 3, \ldots$. Consequently (see condition 2 of the theorem),

$$V^{\bullet}(\tau, \mathbf{x}(\tau; t_0, \mathbf{x}_*)) \leqslant -l/2 \quad \text{for} \quad \tau \in [t_k - \beta, t_k + \beta], \quad k = 1, 2, 3, \ldots$$

Further, we have

$$0 \leqslant V(t_k + \beta, \mathbf{x}(t_k + \beta; t_0, \mathbf{x}_*)) = V(t_0, \mathbf{x}_*) + \int_{t_0}^{t_k+\beta} V^{\bullet}(\tau, \mathbf{x}(\tau; t_0, \mathbf{x}_*))\, d\tau \leqslant$$

$$\leqslant V(t_0, \mathbf{x}_*) + \sum_{i=1}^{k} \int_{t_i-\beta}^{t_i+\beta} V^{\bullet}(\tau, \mathbf{x}(\tau; t_0, \mathbf{x}_*))\, d\tau \leqslant V(t_0, \mathbf{x}_*) - l\beta k$$

which is impossible if k is sufficiently large. The theorem is proved.

Theorem 2. If a function $V(t,\ \mathbf{x})$ exists, satisfying conditions (1) and (2) of Theorem 1, and $W^{\bullet} \leqslant 0$, then the motion $\mathbf{x} = 0$ is asymptotically y-stable. If, furthermore, system (1.1) and function W are ω-periodic in t (or do not depend on time), then the asymptotic $\mathbf{y}$-stability is uniform in $\{t_0, \mathbf{x}_0\}$.

Proof. From the theorem's hypotheses it follows that

$$W(t, \mathbf{x}(t; t_0, \mathbf{x}_0)) \downarrow 0 \quad \text{as} \quad t \to \infty \tag{1.5}$$

Otherwise, $W(t, \mathbf{x}(t; t_0, \mathbf{x}_0)) \geqslant l > 0$ for $t \geqslant t_0$. But then $V^{\cdot}(t, \mathbf{x}(t; t_0, \mathbf{x}_0)) \leqslant -l$ for $t \geqslant t_0$, and from relation

$$V(t, \mathbf{x}(t; t_0, \mathbf{x}_0)) = V(t_0, \mathbf{x}_0) + \int_{t_0}^{t} V^{\cdot}(\tau, \mathbf{x}(\tau; t_0, \mathbf{x}_0))\, d\tau$$

follows

$$V(t, \mathbf{x}(t; t_0, \mathbf{x}_0)) \leqslant V(t_0, \mathbf{x}_0) - l(t - t_0)$$

which is impossible if t is sufficiently large. From (1.5) we conclude that motion $\mathbf{x} = 0$ is asymptotically $\mathbf{y}$-stable. When system (1.1) and function W are ω-periodic in t, the required uniformity follows from Theorem 1 of [3].

Note. The theorems proved generalize, respectively, Theorems 3 and 4 of [3].

2. The motion of a holonomic mechanical system with generalized coordinates $q_1, \ldots, q_n$ and time-independent constraints, which is under the action of potential, gyroscopic, and dissipative forces, is described by a system of Lagrange equations

$$\frac{d}{dt}\frac{\partial T}{\partial q_i^{\cdot}} - \frac{\partial T}{\partial q_i} = -\frac{\partial U}{\partial q_i} + \sum_{j=1}^{n} g_{ij} q_j^{\cdot} - \frac{\partial f}{\partial q_i^{\cdot}} \quad (i = 1, \ldots, n),\; g_{ij} = -g_{ji} \tag{2.1}$$

Taking the total energy $H = T + U$ of the system as the Liapunov function, we obtain [2]

$$H^{\cdot} = -2f \tag{2.2}$$

Example 1 in [3] admits of the following generalization. Assume that:

1) system (2.1) admits of a particular solution $\mathbf{q} = \mathbf{q}^{\cdot} = \mathbf{0}$ (the equilibrium position);

2) the potential energy $U = U(q_1, \ldots, q_n)$ is positive definite in $q_1, \ldots, q_m$ $(m < n)$, while the dissipative function $f = f(q_1^{\cdot}, \ldots, q_n^{\cdot})$ is a quadratic form positive definite relative to all the velocities, and by virtue of (2.2), the equilibrium position is stable with respect to $q_1, \ldots, q_m, q_1^{\cdot}, \ldots, q_n^{\cdot}$ [2];

3) from some mechanical considerations it is known that each solution of system (2.1), located in some neighborhood of the point $\mathbf{q} = \mathbf{q}^{\cdot} = \mathbf{0}$, is bounded (*);

4) there are no equilibrium positions in the set $U(\mathbf{q}) > 0$.

By repeating the arguments in Example 1 of [3], with the obvious replacement of the set $q_1^2 + \ldots + q_m^2 > 0$ by the set $U(\mathbf{q}) > 0$, we convince ourselves that $H(\mathbf{q}(t), \mathbf{q}^{\cdot}(t)) \downarrow 0$ as $t \to \infty$ if $\|\mathbf{q}(0)\| + \|\mathbf{q}^{\cdot}(0)\|$ is sufficiently small. By Theorem 1 in [3], from this it follows that the equilibrium position $\mathbf{q} = \mathbf{q}^{\cdot} = \mathbf{0}$ is asymptotically stable relative to $q_1, \ldots, q_m, q_1^{\cdot}, \ldots, q_n^{\cdot}$ uniformly in $\{t_0, \mathbf{q}_0, \mathbf{q}_0^{\cdot}\}$. In such a form this example can be extended to the stability problem for the equilibrium position of a solid body with a liquid filling.

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*) This condition can be taken as fulfilled, and all the subsequent conclusions remain in force, for example, if the coordinates $q_{m+1}, \ldots, q_n$ are angular (mod 2π) and all the quantities occurring in system (2.1) and the function H are 2π-periodic in $q_{m+1}, \ldots, q_n$ [3].

3. Let us now consider a solid body having a simply-connected cavity which is partially or wholly filled with a homogeneous incompressible viscous (or ideal) liquid. We assume that the constraints imposed are independent of time and that potential forces as well as external dissipative forces with complete dissipation act on the system; we neglect the surface tension force. We take the equations of motion in the form [4]

$$\frac{d}{dt}\frac{\partial T}{\partial q_j^{\cdot}} - \frac{\partial T}{\partial q_j} = \Phi_j - \frac{\partial f}{\partial q_j^{\cdot}} \qquad (j = 1, \ldots, n \leqslant 6) \tag{3.1}$$
$$\frac{d\mathbf{v}}{dt} + \boldsymbol{\omega} \times \mathbf{v} = \mathbf{F} - \frac{1}{\rho}\operatorname{grad} p + \nu\Delta\mathbf{v}, \quad \operatorname{div}\mathbf{v} = 0$$

To these equations we should add on the appropriate boundary and initial conditions; here $f(q_1^{\cdot}, \ldots, q_n^{\cdot})$ is a positive-definite quadratic form, $\nu > 0$ for a viscous liquid and $\nu = 0$ for an ideal liquid.

By virtue of the assumption that the forces acting are potential, we have [4]

$$\sum_{j=1}^{n} \Phi_j q_j^{\cdot} + \rho \int_{\tau} \mathbf{F}\cdot\mathbf{u}\,d\tau = -U^{\cdot}$$

where $\mathbf{u}$ is the relative velocity vector. Taking the system's total energy $H = T + U$ as the Liapunov function, we obtain [4]

$$H^{\cdot} = -2f - \int_{\tau} E\,d\tau \tag{3.2}$$

here [4]

$$E = \mu\left\{2\sum_{i=1}^{3}\left(\frac{\partial v_i}{\partial x_i}\right)^2 + \left(\frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2}\right)^2 + \left(\frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3}\right)^2 + \left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right)^2\right\}$$

and $x_1x_2x_3$ is a moving coordinate system rigidly fixed to the body. By $e_{ij} = {}^1/_2\,(\partial v_i / \partial x_j + \partial v_j / \partial x_i)$ we denote the components of the deformation rate tensor. Further, as in [5], we make certain assumptions on the nature of the perturbed motion. Following [5] we make a continuous change of variables

$$\lambda = \lambda(x_1, x_2, x_3), \quad \nu = \nu(x_1, x_2, x_3), \quad \tau = W(x_1, x_2, x_3)$$

and we let the equation of the side wall be $\nu(x_1, x_2, x_3) = \beta_0 = \text{const}$, while we represent the equation of the liquid's free surface in the form $\tau - \alpha_0 = \varkappa(t, \lambda, \nu)$, $\alpha_0 = \text{const}$. We consider that by specifying the values

$$q_0,\ q_0^{\cdot},\ u_i(t_0, x_1, x_2, x_3) = \varphi_i(x_1, x_2, x_3) \qquad (i = 1, 2, 3)$$
$$\varkappa(t_0, \lambda, \nu)$$

at the initial instant $t = t_0$ and specifying $\operatorname{div}\mathbf{u} = 0$, the subsequent motion of the system is uniquely determined.

We assume that (Cf. Sect. 2):

1) the equations of motion (3.1) admit a particular solution $\mathbf{q} = \mathbf{q}^{\cdot} = 0$, $\mathbf{v} = 0$ (the equilibrium position);

2) the potential energy U is positive definite in $q_1, \ldots, q_m$ $(m < n)$; and by virtue of (3.2), the equilibrium position is stable relative to $q_1, \ldots, q_m, q_1^{\cdot}, \ldots, q_n^{\cdot}, T_2$ (T_2 is the liquid's kinetic energy);

3) the coordinates $q_{m+1}, \ldots, q_n$ are cyclic (mod 2π), and all the quantities occurring in system (3.1) and the function H are 2π-periodic in $q_{m+1}, \ldots, q_n$; nere

we can take it [3] that $q_{m+1}(t), \ldots, q_n(t)$ are bounded in the perturbed motion;

4) there are no equilibrium positions in the set $U > 0$.

Suppose that in the perturbed motion the deviation ∇ satisfies the condition [4] $\nabla > \varepsilon l$ for every $t \geqslant t_0 \geqslant 0$. We accept the following assumptions. Assumption–

A) Along every perturbed motion the liquid's relative velocity vector $\mathbf{u}$ and its total time derivative $\mathbf{u}^{\cdot}$ remain uniformly bounded throughout the liquid during the entire motion,

$$\|\mathbf{u}\| \leqslant M, \quad \|\mathbf{u}^{\cdot}\| \leqslant M, \quad M = \text{const} > 0$$

In such a case the derivative $f^{\cdot}$ relative to the first group of Eqs. (3.1), solved with respect to $q_j^{\cdot\cdot}$ $(j = 1, \ldots, n)$, is bounded along any perturbed motion. Hence, taking (3.2) into account, we conclude on the basis of Theorem 1 that the equilibrium position is asymptotically stable relative to $q_1^{\cdot}, \ldots, q_n^{\cdot}$ (*). This conclusion is valid both for a viscous as well as for an ideal liquid. From now on we assume that the liquid is viscous.

Together with Assumption A we accept the following Assumptions B, C and D.

B) Along any perturbed motion the components of the deformation rate tensor and their total time derivatives and partial derivatives with respect to coordinates x_s, as well as the partial derivatives $\partial u_i / \partial x_j$, remain uniformly bounded throughout the liquid during the entire motion,

$$|e_{ij}| \leqslant M, \quad |e_{ij}^{\cdot}| \leqslant M, \quad \left|\frac{\partial e_{ij}}{\partial x_s}\right| \leqslant M, \quad \left|\frac{\partial u_i}{\partial x_j}\right| \leqslant M \quad (i, j, s = 1, 2, 3) \tag{3.3}$$

C) [5] The function $\varkappa(t, \lambda, \nu)$ is continuous in λ, ν uniformly in $t \geqslant 0$, i.e. for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that from $|\lambda' - \lambda''| < \delta$, $|\nu' - \nu''| < \delta$ follows $|\varkappa(t, \lambda', \nu') - \varkappa(t, \lambda'', \nu'')| < \varepsilon$ for all $t \geqslant 0$.

D) [5] The function H depends continuously on the initial conditions, i.e. for any $\varepsilon > 0$, $\theta > 0$ there exists $\delta(\varepsilon, \theta) > 0$ such that from

$$\|q_0' - q_0''\| < \delta, \quad \|q_0'^{\cdot} - q_0''^{\cdot}\| < \delta, \quad |\varphi_i'(x_1, x_2, x_3) - \varphi_i''(x_1, x_2, x_3)| < \delta$$
$$|\varkappa'(0, \lambda, \nu) - \varkappa''(0, \lambda, \nu)| < \delta$$

follows

$$|H(\mathbf{q}'(\theta), \mathbf{q}'^{\cdot}(\theta), \mathbf{u}'(\theta, x_1, x_2, x_3), \varkappa'(\theta, \lambda, \nu)) -$$
$$H(\mathbf{q}''(\theta), \mathbf{q}''^{\cdot}(\theta), \mathbf{u}''(\theta, x_1, x_2, x_3)\ \varkappa''(\theta, \lambda, \nu))| < \varepsilon$$

By virtue of (3.3), $E^{\cdot}$ and

$$\frac{d}{dt}\int_{\tau} E\, d\tau$$

are bounded in the perturbed motion. Therefore, in the perturbed motion $e_{ij} \to 0$ as $t \to \infty$ throughout the liquid (this is proved with the use of (3.2) in just the same way as Theorem 1). Furthermore, as was established, $\mathbf{q}^{\cdot}(t) \to 0$ as $t \to \infty$. Consequently, the kinetic energy $T \to 0$ as $t \to \infty$.

Let us show that $H \to 0$ as $t \to \infty$. Assume the contrary; then $H \to H^* > 0$ and

$$H \geqslant H^* \quad \text{for} \quad t \geqslant t_0 \tag{3.4}$$

for any sequence of $t_s \to \infty$ the sequence of functions $\{\varkappa(t_s, \lambda, \nu)\}$ is uniformly bounded and equicontinuous (see C), therefore, by virtue of the Arzela theorem, from it we can

*) The system being considered has an infinite number of degrees of freedom. However, the proof of Theorem 1 in this case is completely preserved.

single out a convergent subsequence. Thus, for some sequence of $t_k \to \infty$,

$$\mathbf{q}(t_k) \to \mathbf{q}_*, \quad \mathbf{q}^{\cdot}(t_k) \to 0, \quad \mathbf{u}(t_k, x_1, x_2, x_3) \to 0, \varkappa(t_k, \lambda, \nu) \to \varkappa_*(\lambda, \nu)$$

We note that

$$H|_{\mathbf{q}=\mathbf{q}_*, \mathbf{q}^{\cdot}=0, \mathbf{u}=0, \varkappa=\varkappa_*} = H^*$$

consequently, $U(\mathbf{q}_*, \varkappa_*) = H^* > 0$. Taking the limit "point" $(\mathbf{q}_*, \mathbf{q}_*^{\cdot} = 0, \mathbf{u} = 0, \varkappa_*)$ as the origin of a new perturbed motion $(\mathbf{q}^*(t), \mathbf{q}^{*\cdot}(t), \mathbf{u}^*(t, x_1, x_2, x_3), \varkappa^*(t, \lambda, \nu))$, we obtain by virtue of Assumption 4 that for some $\theta > 0$.

$$H(\mathbf{q}^*(\theta), \mathbf{q}^{*\cdot}(\theta), \mathbf{u}^*(\theta, x_1, x_2, x_3), \varkappa^*(\theta, \lambda, \nu)) < H^*$$

Using the continuous dependency (Condition D) and the group property of autonomous systems, just as in Example 1 of [3], we obtain that in the original perturbed motion,

$$H(\mathbf{q}(t_k + \theta), \mathbf{q}^{\cdot}(t_k + \theta), \mathbf{u}(t_k + \theta, x_1, x_2, x_3)\, \varkappa(t_k + \theta, \lambda, \nu)) < H^*$$

for all k greater than some N, which contradicts inequality (3.4).

Thus, in the perturbed motion, $H \to 0$, $\mathbf{q}^{\cdot} \to 0$, $q_i \to 0$ $(i = 1, \ldots, m)$, $\mathbf{u} \to 0$, as $t \to \infty$. Let $\alpha > 0$ be such that from

$$\|\mathbf{q}_0\| \leqslant \alpha, \ \|\mathbf{q}_0^{\cdot}\| \leqslant \alpha, \ |\varphi_i(x_1, x_2, x_3)| \leqslant \alpha \quad (i = 1, 2, 3)$$
$$|\varkappa(0, \lambda, \nu) - \varkappa^\circ(\lambda, \nu)| \leqslant \alpha \tag{3.5}$$

where $\tau - \alpha_0 = \varkappa^\circ(\lambda, \nu)$ is the equation of the free surface in the unperturbed motion, follows $H \to 0$ as $t \to \infty$. Among the functions φ_i and $\varkappa(0, \lambda, \nu)$ satisfying condition (3.5) we select a class K of equicontinuous ones (by virtue of (3.5) these functions are uniformly bounded). Let us show that the relation $H \to 0$ as $t \to \infty$ is fulfilled uniformly with respect to initial perturbations from the "domain"

$$\|\mathbf{q}_0\| \leqslant \alpha, \ \|\mathbf{q}_0^{\cdot}\| \leqslant \alpha, \ (\varphi_i, \varkappa) \in K \quad (i = 1, 2, 3) \tag{3.6}$$

i.e. for any $\varepsilon > 0$ there exists $\theta(\varepsilon) > 0$ such that if at $t = 0$ the initial perturbations lie in domain (3.6), then $H < \varepsilon$ for all $t \geqslant 0$.

Indeed, for any point $\mu_0 = (\mathbf{q}_0, \mathbf{q}_0^{\cdot}, \varphi_1, \varphi_2, \varphi_3, \varkappa)$ from domain (3.6) there exists $\theta(\varepsilon, \mu_0) > 0$ such that

$$H(\mu(\theta, \mu_0)) < \varepsilon \tag{3.7}$$

Here $\mu(t, \mu_0)$ denotes the solution $(\mathbf{q}(t), \mathbf{q}^{\cdot}(t), \mathbf{u}(t), \varkappa(t, \lambda, \nu))$ issuing from the point μ_0 at $t = 0$. By continuity (Condition D) there exists a neighborhood $O(\mu_0)$ of point μ_0 such that for any point $\mu_0' \in O(\mu_0)$

$$H(\mu(\theta, \mu_0')) < \varepsilon \tag{3.8}$$

H decreases monotonically along the motion, therefore, from (3.8) it follows that

$$H(\mu(t, \mu_0')) < \varepsilon \quad \text{for} \quad t \geqslant \theta, \ \mu_0' \in O(\mu_0) \tag{3.9}$$

Uniformly bounded and equicontinuous functions occur in class K, therefore, domain (3.6) is compact and, consequently, from the system $\{O(\mu_0)\}$ of neighborhoods covering it we can single out a finite subcover $O_1, \ldots, O_s$. Let the numbers $\theta_1, \ldots, \theta_s$ correspond to this subcover. We set $\theta(\varepsilon) = \max\{\theta_1, \ldots, \theta_s\}$ ($\theta(\varepsilon)$ depends only on ε). From (3.9) it follows that

$$H(\mu(t, \mu_0)) < \varepsilon \quad \text{for} \ t \geqslant \theta(\varepsilon)$$

if μ_0 lies in domain (3.6). Q.E.D.

Consequently, the equilibrium position is asymptotically stable relative to $q_1, \ldots, q_m, q_1^{\cdot}, \ldots, q_n^{\cdot}$, T_2 uniformly with respect to initial conditions from domain (3.6).

Note. It is easy to show analogously [5] that in the case being considered here the dissipative forces possess partial dissipation or are entirely absent ($f \equiv 0$), while in the set $H > 0$ there are no motions of the whole system as a single solid body (see Zhukovskii's theorem in [4] p. 67), then in the perturbed motion $H \to 0$ as $t \to \infty$ and, what is more, uniformly in domain (3.6), which is proved analogously to the above. Consequently, the conclusion on asymptotic stability relative to $q_1, \ldots, q_m, q_1^{\cdot}, \ldots, q_n^{\cdot}$, T_2, uniform with respect to initial conditions from domain (3.6), remains in force. From this, in the special case when the potential energy U has a minimum at the equilibrium position, there follows an addition to Theorem 1.1 of [5] concerning uniformity with respect to initial conditions from domain (3.6).

The author thanks V.V. Rumiantsev for guidance and constant attention.

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Translated by N.H.C.

UDC 62-50

APPROXIMATE SYNTHESIS METHOD FOR OPTIMAL CONTROL OF A SYSTEM SUBJECTED TO RANDOM PERTURBATIONS

PMM Vol. 36, №5, 1972, pp. 829-839

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(Received November 18, 1971)

An approximate method is proposed for synthesizing the optimal control for a dynamical system in the presence of external random perturbations and measurement errors. The synthesis problem posed reduces, as is known, to solving a nonlinear parabolic partial differential equation (the Bellman equation) whose exact solutions are known only in a few cases. It is assumed that either the external perturbations acting on the system are sufficiently small or the